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A Nonmonotonic Theory of Probability for Spin-$\frac{1}{2}$ Systems

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Kolmogorov's theory of probability is monotonic, meaning that the probability of A is less than or equal to the probability of B whenever A entails B. A nonmonotonic theory of probability is obtained, if the greatest lower bound for probabilities is set at -1 instead of 0, the value fixed by Kolmogorov's positivity axiom. The new theory retains Kolmogorov's other axioms, and many important theorems still hold. It also has substantial applicability: it can accommodate probabilities for spin-$\frac{1}{2}$ systems while preserving Boolean operations. That is to say, negative probabilities are here provided with a homely setting in the quantum domain.

KEY WORDS: quantum mechanics; foundations; probability; spin.

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1. A NONMONOTONIC THEORY OF PROBABILITY

It is not possible to preserve Boolean operations within the confines of the standard theory of probability while accommodating probabilities predicted by the formalism of quantum mechanics. Non-Boolean logics have been explored extensively in the context of the foundations of quantum mechanics as a result. Substantial motivation for considering a nonmonotonic theory of probability is provided here by showing that it can accommodate probabilities for spin-$\frac{1}{2}$ systems while preserving Boolean operations.

Let $C[S]$ denote the closure of a finite set S of basic propositions with respect to $\vee$, $\wedge$, $\neg$; that is to say, the Boolean operators that respectively correspond to *or*, *and*, *not*. The axioms of a nonmonotonic theory of probability, which hold for any A, B $\in C[S]$, are the following:

Extension:

$$P(\mathrm{A}) \geq -1$$

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Total probability:

$$P(\text{A}) = 1, \quad \text{if A is a tautology}$$

Conditional probability:

$$P(\text{A}|\text{B}) = P(\text{A} \wedge \text{B})/P(\text{B}), \quad \text{if } P(\text{B}) \neq 0$$

Additivity:

$$P(\text{A} \vee \text{B}) = P(\text{A}) + P(\text{B}), \quad \text{if A} \mapsto \neg\text{B}$$

The first replaces Kolmogorov's (1950) *positivity* axiom, $P(\text{A}) \geq 0$, the rest correspond to his other axioms excluding *countable-additivity*. It can be shown that the new axiom set is consistent, that its elements are logically independent, and that it entails many of the theorems that can be derived using the standard axiom set.[2]. Three of these theorems will be used below and they are the following:

Negation:

$$P(\neg\text{A}) = 1 - P(\text{A})$$

Equivalence:

$$P(\text{A}) = P(\text{B}), \quad \text{if A} \mapsto \text{B} \quad \text{and} \quad \text{B} \mapsto \text{A}$$

General additivity:

$$P(\text{A} \vee \text{B}) = P(\text{A}) + P(\text{B}) - P(\text{A} \wedge \text{B})$$

One interesting and important theorem of the standard axiom set *fails* to hold in the new set. This theorem is the following:

monotonicity

$$P(\text{A}) \leq P(\text{B}), \quad \text{if A} \mapsto \text{B}$$

Thus, it is appropriate to refer to this theory as a *nonmonotonic* probability theory.

The strategy of inducing a new formal system by tweaking one or more of the axioms of the standard system is familiar in the study of quantum structures. Quantum logic was obtained by replacing the distributive law of classical logic with ortho-modularity (Loomis, 1955).[3] The resulting system was explored extensively for several decades in the foundations of quantum mechanics. It did not ultimately lead to a realistic interpretation of quantum mechanics as some of its proponents

[2] Proofs of consistency and independence can be developed from Appendix *iv of (Popper, 1968); see his model on p. 343 and note that his axioms B1 and A4′ correspond to *Monotonicity* and *Positivity*, respectively. Proofs of the usual theorems (including those mentioned below) are provided in Chapter 2 of (Howson and Urbach, 1993). The axioms are presented in (Kolmogorov, 1950).

[3] There are at least two other notable systems of quantum logics. One involves replacing distributivity with modularity (Birkhoff and von Neumann, 1936), the other replaces bi-valence with tri-valence (Reichenbach, 1948).

had hoped, but it did result in a deeper understanding of the logical structure of quantum mechanics. (For an excellent summary of these developments, see Foulis, 1999).

There are non-Kolmogorovian probability theories that invoke number fields other than the reals. Gudder (1988) has developed a quantum probability theory over the complex numbers. Khrennikov (1999) has developed a p-adic probability theory in which limits of relative frequencies are determined using a nonstandard metric. The p-adic number field is a type of non-Archimedean field. The nonmonotonic theory that is under consideration here is much more modest than the two just mentioned in that it is over the real number field.

Complications arise when one tries to do theoretical justice to probabilities obtained experimentally, meaning in actual spin measurements. There is strong evidence suggesting that such situations require a generalization of quantum mechanics. One viable approach is to replace projection-valued measures, the standard textbook treatment of spin and other observables, with positive operator-valued (POV) measures. Associated with the use of POV measures for spin is the thesis of the "unsharp reality" of spin (Busch and Schroeck, 1989). It is a provocative and well-argued position, so it is of substantial interest to determine whether the nonmonotonic framework can handle a more realistic and sophisticated quantum mechanical characterization of probabilities obtained in actual spin measurements. This matter is beyond the scope of the current paper.

2. NEGATIVE PROBABILITIES IN THE TWO-SLIT EXPERIMENT

The two-slit experiment is widely used in discussions of the foundations of quantum mechanics. It is easily discussed qualitatively, making it a good starting place to consider negative probabilities in quantum interference, but a quantitative treatment of it within quantum mechanics is quite technical. It is much better to consider quantum interference phenomena involving spin-$\frac{1}{2}$ systems, since they are governed by a simple $\cos^2(\theta/2)$ law. A preliminary qualitative treatment of the two-slit experiment is provided in this section to graphically motivate negative probabilities. This is done using the standard portrayal of the two-slit experiment in terms of probabilities. Three alternative portrayals are then introduced. It turns out in light of the discussion of spin-$\frac{1}{2}$ systems below that one of the three alternative portrayals is better than the other two as well as the original.

In Figs. 1–3, A denotes the upper slit, B the lower slit, C the point on the detection screen of maximal constructive interference. In what follows, X denotes any arbitrary point on the detection screen. Because context removes ambiguity, A [B] also denotes "the photon passes through slit A [B]," and C [X] denotes "the photon is detected at C [X]." In typical portrayals of the experiment it is assumed that $P(\mathrm{A} \wedge \mathrm{X})$ is obtained by closing off slit B (Fig. 1), $P(\mathrm{B} \wedge \mathrm{X})$ by closing off slit A, and $P((\mathrm{A} \vee \mathrm{B}) \wedge \mathrm{X})$ by having both open. It is then noted that if photons

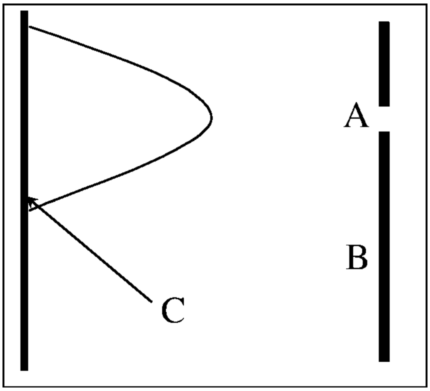

Fig. 1. A: open, B: closed.

were particles, $P((\mathrm{A} \vee \mathrm{B}) \wedge \mathrm{X})$ would exhibit the additive pattern (Fig. 2), the sum of $P(\mathrm{A} \wedge \mathrm{X})$ and $P(\mathrm{B} \wedge \mathrm{X})$, but that what is actually obtained is the interference pattern (Fig. 3).

The usual conclusion drawn is that photons do not behave like classical particles experimentally when passing through the slits (although they do so in their interaction with the detection screen). However, another conclusion seems to follow; namely, that negative probabilities are involved. A comparison of Figs. 2 and 3 reveals that $P((\mathrm{A} \vee \mathrm{B}) \wedge \mathrm{C}) > P(\mathrm{A} \wedge \mathrm{C}) + P(\mathrm{B} \vee \mathrm{C})$, and so by *General additivity* $P(\mathrm{A} \wedge \mathrm{B} \wedge \mathrm{C}) < 0$.

Is the portrayal above of the two-slit experiment in terms of probability correct? To see what is at issue, consider the following questions: Is it $P(\mathrm{A} \wedge \mathrm{X})$ or $P(\mathrm{A} \wedge \neg\mathrm{B} \wedge \mathrm{X})$ that is obtained when slit B is closed off? Similarly, is it $P(\mathrm{B} \wedge \mathrm{X})$ or $P(\neg\mathrm{A} \wedge \mathrm{B} \wedge \mathrm{X})$ when slit A is closed off? Finally, is it $P((\mathrm{A} \vee \mathrm{B}) \wedge \mathrm{X})$ or

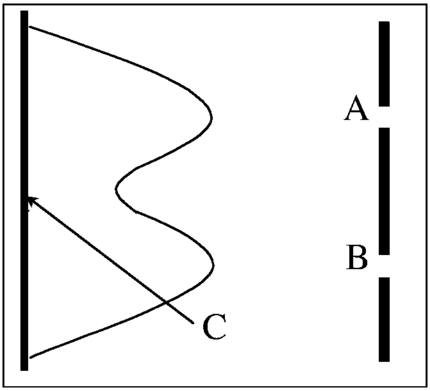

Fig. 2. Additive pattern.

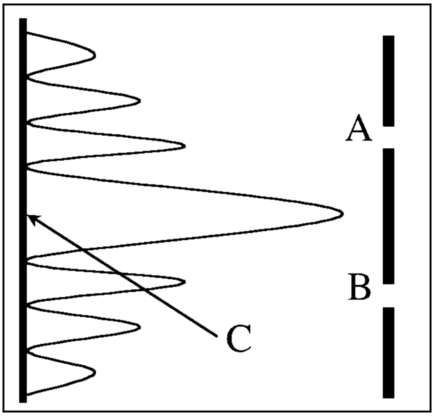

Fig. 3. Interference pattern.

$P(\mathrm{A} \wedge \mathrm{B} \wedge \mathrm{X})$ when both are open? These questions suggest at least four possible portrayals of the three modes of the two-slit experiment, as indicated in Table I.

Only Portrayal 4 is viable, the others have *divergent* conditional probabilities.[4] The viability of Portrayal 4 is demonstrated in the next section. The nonviability of the other three is shown in the Appendix. The upshot for the two-slit experiment is this. The operational probability is $P(\mathrm{A} \wedge \neg\mathrm{B} \wedge \mathrm{X})$ when slit A is open and slit B is closed, $P(\neg\mathrm{A} \wedge \mathrm{B} \wedge \mathrm{X})$ when slit A is closed and slit B open, and $P(\mathrm{A} \wedge \mathrm{B} \wedge \mathrm{X})$ when both slits are open.

3. STERN–GERLACH EXPERIMENTS INVOLVING SPIN-$\frac{1}{2}$ SYSTEMS

A paradigm example of a spin-$\frac{1}{2}$ measurement involves the outermost electron of a silver atom. Let z be the direction of motion of the atom, u be any direction in the xy-plane (the plane perpendicular to z), and let spin-u denote the electron observable 1/2-integral spin in the u-direction. Since there are just two possible outcomes for a spin-u measurement, $+1$ and -1 (in units of $\hbar/2$), the spin-u operator has two eigenvalues, denoted here as $u+$ and $u-$, and their respective eigenvectors, $|u+\rangle$ and $|u-\rangle$. Spin-u is measured as follows. First, the spin-u eigenvalues of the outermost electron of the silver atom are correlated with the position of the atom's center of mass along the u-axis. Passing the atom through an inhomogeneous magnetic field that is oriented in the u-direction creates the correlation. When the atom emerges from the field it is possible to measure spin-u

[4] It is quite possible that the divergence is due to the unphysical idealization of sharp measurements, and that it does not occur in the case of un-sharp measurements. If so, then it may well be that one of the other three portrayals (i.e., other than portrayal 4) may be more appropriate for such measurements. I wish to thank one of the anonymous referees for the *International Journal of Theoretical Physics* for this suggestion.

Table I. Four Portrayals

Modes	Portrayal 1	Portrayal 2	Portrayal 3	Portrayal 4
A open, B closed	A	A	$A \wedge \neg B$	$A \wedge \neg B$
A closed, B open	B	B	$\neg A \wedge B$	$\neg A \wedge B$
A open, B open	$A \vee B$	$A \wedge B$	$A \vee B$	$A \wedge B$

indirectly by measuring the position of the atom with respect to the u-axis using a detecting plate; position near the positive u-axis corresponds to $u+$ and position near the negative u-axis to $u-$.

Suppose now that the detecting plate in the experiment above is replaced with a beam stop that only blocks atoms in the lower beam (those that have a position along the negative u-axis). The atoms that are not blocked by the beam stop are now in the eigenstate $|u+\rangle$, meaning that if these atoms were subsequently measured for spin-u the outcome would be the value corresponding to $u+$ with probability 1. More generally if the atoms were subsequently measured for spin-w, where w is an angle θ in the xy-plane from u, then they would yield the outcome $w+$ with probability $\cos^2(\theta/2)$ and $w-$ with probability $\cos^2((\pi + \theta)/2)$, which equals $\sin^2(\theta/2)$.[5]

The experiments that are used to show the applicability of nonmonotonic probabilities are just slightly more complicated than the one above.[6] The required feature is quantum interference. It is brought about in a beam of spin systems using an analyzer loop, which consists of two Stern–Gerlach magnets that have the same length and field strength, are in sequence, and are oppositely oriented. Let z be the beam's direction of motion, and y' be some direction in the plane orthogonal to z. If the first magnet is oriented in the y'-direction, the second is oriented in the $-y'$-direction and the loop is called a *y'-analyzer loop*. The first separates the beam into two sub-beams each corresponding to one of the spin-y' eigenstates; the second bends the beams back to their original line of motion. If both loop channels are open, then the spin state of the systems in the beam exiting the loop is the same at it was when it entered the loop.

In the experimental set up of special interest, a source produces a beam of spin-$\frac{1}{2}$ systems in the eigenstate $|y+\rangle$ moving in the z-direction. The beam enters a y'-analyzer loop, and after exiting the loop the beam elements are measured for spin-y'' (see Fig. 4). The directions y' and y'' are in the xy-plane, which is perpendicular to the direction of motion z; the angle between y and y' is θ and that between y and y'' is ϕ. There are two channels through the loop, one corresponding to $y'+$ and the other to $y'-$. A beam stop may be placed in one of the channels of the loop, meaning that there are three relevant *modes* of the experiment. The

[5] This is for ideal (possibly unrealizable) spin measurements. Experimentally realistic spin measurements may be un-sharp at least to some degree (as noted in Section 1 above). Only the ideal case is considered in this paper.

[6] See Chapters 6–11 of (Feynman *et al.*, 1965) for a more details about these kinds of experiments.

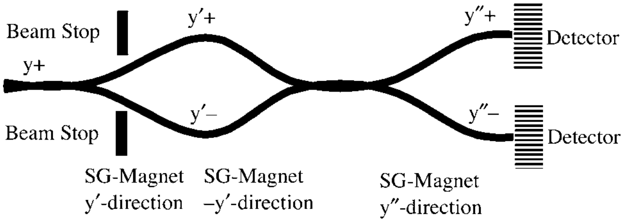

Fig. 4. A y'-analyzer loop in mode-1 followed by a $y''+$ detector.

modes are denoted by the variable χ: both channels open ($\chi = 1$), only the $y'+$ channel open ($\chi = 2$), and only the $y'-$ channel open ($\chi = 3$). An *experimental configuration* is characterized by an ordered triplet $\langle \chi, \theta, \phi \rangle$, which corresponds to particular choices for the mode χ and the *orientation* $\langle \theta, \phi \rangle$, θ for the loop and ϕ for the third magnet. The action of the loop with respect to each of the three modes above is this.

- If both channels are open then the two beams merge together and the emerging beam has the same spin state $|y\rangle$ and intensity I as the entering beam.
- If the $y'-$ channel is blocked, then the emerging beam is in spin state $|y'+\rangle$ and has a reduced intensity $I' = I\ \cos^2(\theta/2)$.
- If the $y'+$ channel is blocked, then the emerging beam is in spin state $|y'-\rangle$ and has a reduced intensity $I' = I\ \sin^2(\theta/2)$.

After a beam emerges from the analyzer loop, it is measured for spin-y'' using a Stern–Gerlach magnet oriented in the y'' direction—the angle between y and y'' is ϕ. Let I'' denote the intensity of the $y''+$ beam that emerges from this magnet. The following results are then obtained.

- If both channels are open, then $I = I \cos^2(\phi/2)$.
- If the $y'-$ channel is blocked, then $I = I' \cos^2((\phi - \theta)/2) = I \cos^2(\theta/2) \cos^2((\phi - \theta)/2)$.
- If the $y'+$ channel is blocked, then $I = I' \cos^2((\phi - \theta + \pi)/2) = I \sin^2 \theta \sin^2((\phi - \theta)/2)$.

It is assumed that I is normalized to unity by choice of suitable units (without loss of generality).

Outcomes of the three modes of the spin-interference experiments under consideration will now be characterized in terms of the probability theory. To facilitate doing so, the key events in the experiment are denoted as follows.

A: The system passes through the $y'+$ channel,
B: The system passes through the $y'-$ channel,
C: The system activates the $y''+$ detector.

The following operational probabilities for the three modes and arbitrary orientation $\langle\theta, \phi\rangle$ summarizes the information presented above.

$$P(\mathrm{A} \wedge \mathrm{B} \wedge \mathrm{C}|\langle 1, \theta, \phi\rangle) = \cos^2(\phi/2)$$

$$P(\mathrm{A} \wedge \neg\mathrm{B} \wedge \mathrm{C}|\langle 2, \theta, \phi\rangle) = \cos^2(\theta/2)\cos^2((\phi - \theta)/2)$$

$$P(\neg\mathrm{A} \wedge \mathrm{B} \wedge \mathrm{C}|\langle 3, \theta, \phi\rangle) = \sin^2(\theta/2)\sin^2((\phi - \theta)/2)$$

A crucial interpretive move is now made by introducing the following two sets of identifications.

$$P(\mathrm{A} \wedge \neg\mathrm{B} \wedge \mathrm{C}|\langle 1, \theta, \phi\rangle) = P(\mathrm{A} \wedge \neg\mathrm{B} \wedge \mathrm{C}|\langle 2, \theta, \phi\rangle)$$
$$P(\neg\mathrm{A} \wedge \mathrm{B} \wedge \mathrm{C}|\langle 1, \theta, \phi\rangle) = P(\neg\mathrm{A} \wedge \mathrm{B} \wedge \mathrm{C}|\langle 3, \theta, \phi\rangle)$$

These identifications involve the supposition that probabilities obtained operationally in mode 2 and mode 3 are significant from an interpretive point of view for the corresponding events in mode 1 where quantum interference occurs. That is, probabilities that are obtained operationally (quantum measured relative frequencies) in one context (mode 2 or mode 3) may be carried over and assigned to another (mode 1) in which they cannot be obtained operationally. Such probabilities are associated with virtual events, and may be thought of ontologically speaking as characterizing strengths of probabilistic propensities. This interpretive move is reasonable, but its justification will be provided elsewhere. To simplify the equations above the terms C and $\langle 1,\theta,\phi\rangle$ are suppressed. The three key equations for mode 1 are then expressed as follows.

$$P(\mathrm{A} \wedge \mathrm{B}) = \cos^2(\phi/2)$$

$$P(\mathrm{A} \wedge \neg\mathrm{B}) = \cos^2(\theta/2)\ \cos^2((\phi - \theta)/2)$$

$$P(\neg\mathrm{A} \wedge \mathrm{B}) = \sin^2(\theta/2)\ \sin^2((\phi - \theta)/2)$$

The marginals, P(A) and P(B), can be derived from these equations using *Equivalence*, since A is logically equivalent to $(\mathrm{A} \wedge \mathrm{B}) \vee (\mathrm{A} \wedge \neg\mathrm{B})$ and similarly for B. The marginals can then be used to obtain the conditionals, $P(\mathrm{A}|\mathrm{B})$ and $P(\mathrm{B}|\mathrm{A})$, using *Conditional probability*. The disjunctives $P(\mathrm{A} \vee \mathrm{B})$ also follow from the equations above using *Equivalence* since $\mathrm{A} \vee \mathrm{B}$ is logically equivalent to $(\mathrm{A} \wedge \mathrm{B}) \vee (\mathrm{A} \wedge \neg\mathrm{B}) \vee (\neg\mathrm{A} \wedge \mathrm{B})$. Thus, one obtains the following.

$$P(\mathrm{A}) = \cos^2(\phi/2) + \cos^2(\theta/2)\ \cos^2((\phi - \theta)/2)$$

$$P(\mathrm{B}) = \cos^2(\phi/2) + \sin^2(\theta/2)\ \sin^2((\phi - \theta)/2)$$

$$P(\mathrm{A}|\mathrm{B}) = \cos^2(\phi/2)/(\cos^2(\phi/2) + \sin^2(\theta/2)\ \sin^2((\phi - \theta)/2))$$

$$P(\mathrm{B}|\mathrm{A}) = \cos^2(\phi/2)/(\cos^2(\phi/2) + \cos^2(\theta/2)\ \cos^2((\phi - \theta)/2))$$

$$P(\mathrm{A} \vee \mathrm{B}) = \cos^2(\phi/2) + \cos^2(\theta/2)\ \cos^2((\phi-\theta)/2) + \sin^2(\theta/2)\ \sin^2((\phi-\theta)/2)$$

Table II. Interference Involving Spin-$\frac{1}{2}$ States, for $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq \pi$

	Range	Min at	Max at
P(A)	[0, 2]	$(0, \pi), (\pi, \pi)$	(0, 0)
P(B)	[0, 2]	$(0, \pi), (\pi, \pi)$	$(\pi, 0)$
P(A|B)	[0, 1]	(θ, π)	$(0, \phi)$
P(B|A)	[0, 1]	(θ, π)	(π, ϕ)
P(A∨B)	[0, 2]	$(0, \pi), (\pi, \pi)$	$(0, 0), (\pi, 0)$

P(A), P(B), and P(A ∨ B) range from 0 to 2. For P(A) the minimum occurs at $(0, \pi)$ and (π, π), the maximum at (0, 0). For P(B) the minimum also occurs at $(0, \pi)$ and (π, π), but the maximum at $(\pi, 0)$. For P(A ∨ B) the minimum occurs at $(\pi, 0)$ and $(0, \pi)$, and the maximum at (0, 0) and (π, π). P(A|B) and P(B|A) range from 0 to 1. For P(A|B) the minimum occurs at (θ, π) for $0 \leq \theta \leq \pi$ and the maximum at $(0, \phi)$ for $0 \leq \phi < \pi$. For P(B|A) the minimum occurs at (θ, π) for $0 \leq \theta \leq \pi$, the maximum at (π, ϕ) for $0 \leq \phi \leq \pi$.[7] These results are summarized in Table II.

It is noteworthy that the conditionals are within the usual range for Kolmogorovian probabilities, but that the marginals and disjunctives have double the usual range. Since the theorem *Negation* holds for nonmonotonic probabilities, it follows that the range for inferred probabilities in the spin-$\frac{1}{2}$ experiments is from -1 to 2.[8]

4. INTERPRETIVE ISSUES

A crucial move is made in going from operational probabilities to inferred probabilities: the probability that a spin-$\frac{1}{2}$ system passes through one channel and not the other (and then activates the detector) when both paths through the analyzer are open (mode 1) is equated with the probability that is measured when that channel is open and the other is blocked (mode 2 or 3). This move is characterized above as *interpretive* since the terms *event* and *probability* are being used in two distinct senses. In contexts where a channel is blocked, the event either actually occurs or it does not, and the probability is operational in that it is measured as a relative frequency of occurrence. In the context where both channels are open, the event in question (passing through one channel and not the other) is regarded

[7] P(A|B) and P(B|A) are discontinuous at $(0, \pi)$ and (π, π). For $(0, \pi)$: P(A|B) $\to 0$ as $\theta \to 0$, P(A|B) $\to 1$ as $\phi \to \pi$; P(B|A) $\to 0$ as $\theta \to 0$, P(B|A) $\to 1/2$ as $\phi \to \pi$. For (π, π): P(A|B) $\to 0$ as $\theta \to \pi$ and P(A|B) $\to 1/2$ as $\phi \to \pi$; P(B|A) $\to 0$ as $\theta \to \pi$, P(B|A) $\to 1$ as $\phi \to \pi$.

[8] On the contrary, I show that the ranges for the inferred probabilities of interest (marginals, conditionals, disjunctives) in the general case of elliptical polarization is also -1 to 2; the corresponding minimum and maximum values are obtained by dividing the coordinate values in Table II above by 2.

as a virtual event rather than actual, and the associated probability is regarded as inferred rather than operational compare Feynman (1987).

It would be misleading to characterize the key move in the following manner: One must assume that when both analyzer branches are open, photons pass through the branches with rates (or frequencies) the same as what detection-frequencies in the blocked-branch contexts indicate. This characterization is misleading because it underplays the role of two key distinctions: actual versus virtual events, and operational versus inferred probabilities. The move is perhaps best characterized as this: Probabilities for the occurrence of a *virtual* event, meaning one that cannot be directly measured in a given configuration of an experimental set-up, are identified with probabilities obtained for the corresponding *actual* event in another configuration where they can be directly measured. Of course, this complicates matters from an ontological point-of-view since we are now committed to both actual and virtual events. But a more robust ontology such as this is worth considering, if it might serve to capture something very deep and important about the quantum realm. That is to say, it is quite reasonable to include virtual events in one's ontology, if doing so serves to reveal a very interesting and unusual abstract structure lying beneath the surface phenomena. That is the case here. The abstract structure that is revealed by (in effect) including virtual events in one's ontology is a nonmonotonic probability theory.

The interpretive considerations introduced above are both minimal and provisional. A suitable interpretive framework for a nonmonotonic probability theory is to be specified much more fully elsewhere. It is fitting to say a bit more about how this might be pursued. Popper (1968) and Mellor (1971) provide engaging and provocative accounts of operational probabilities in terms of probabilistic propensities. It seems that these accounts could provide the right sort of inspiration for developing the desired framework, the association of inferred probabilities with a new type of probabilistic propensity, a virtual propensity. That is to say, an interpretation of a nonmonotonic probability theory might be developed that is based on a distinction between actual and virtual propensities. Of course, such a propensity interpretation for nonmonotonic probability would need to provide an account of virtual propensities that explains what it means for a virtual event to have a probability that is greater than 1 or less than 0 as well as an account of the significance of the failure of monotonicity.

Finally, it is worth mentioning that the term "interpretive" is not meant to suggest that nonoperational probabilities and their associated virtual events are unphysical. On the contrary, their physical significance is strongly analogous to the physical significance of virtual photons and their associated probabilities in the Gupta–Bleuler framework for quantum electrodynamics (involving an indefinite metric). One might be inclined to draw an analogy between the analysis of spin measurements developed above and counterfactual analyses of EPR-type experiments in which polarizer setting are changed at the wings while considering counterfactually the outcomes for other polarizer settings. However, the analogy with quantum electrodynamics is more robust than the analogy with counterfactual discussions of EPR experiments. Negative probabilities arise in the former but not in the latter. Moreover, there are characterizations of EPR-type experiments involving negative probabilities (Rothman and Sudarshan, 2000), and such characterizations are much more in line with the thesis developed above than characterizations of them involving counterfactuals.

5. SUMMARY

A nonmonotonic theory of probability is formulated in Section 1 that extends the range of probabilities beyond the normal range, the new range being from -1 to 2. The two-slit experiment was briefly considered in Section 2 to demonstrate in a qualitative manner how this theory might be applicable in the quantum domain. The theory was then shown to capture a broad range of interference experiments involving spin-$\frac{1}{2}$ systems in a quantitative manner in Section 3. Some interpretive issues were raised in Section 4. Some tentative stances were adopted in connection with these issues; a much fuller discussion of them will be provided elsewhere. The theory may require modification or generalization to have applicability to N-state systems for $N > 2$. It may also require modification or generalization to handle probabilities obtained in experimentally realistic spin measurements that require the use of positive operator-valued measures. The need for such changes will be discussed much more fully elsewhere.

APPENDIX: DIVERGENT CONDITIONAL PROBABILITIES

Four Portrayals were distinguished in Section 2 (see Table I). In Section 3, probabilities in distinct modes were identified and then used to derive other probabilities using Portrayal 4. A claim was made to the effect that this portrayal is the only viable one of the four. The viability of Portrayal 4 was shown in Section 3. The task of showing the nonviability of the other three was relegated to this appendix. To show this, it suffices to consider cases in which $\phi = 0$ for Portrayals 1 and 2, and $\theta = \pi/2$ for Portrayal 3.

In Portrayal 1 the operational probabilities are these:

$$P((\mathrm{A} \vee \mathrm{B}) \wedge C|\langle 1, \theta, 0\rangle) = 1$$

$$P(\mathrm{A} \wedge \mathrm{C}|\langle 2, \theta, 0\rangle) = \cos^4(\theta/2)$$

$$P(\mathrm{B} \wedge \mathrm{C}|\langle 3, \theta, 0\rangle) = \sin^4(\theta/2)$$

Proceeding by analogy with the treatment of Portrayal 4, two sets of identities are assumed, the terms Z and $\langle 1, \theta, 0\rangle$ are suppressed, and attention is then focused on these probabilities:

$$P(\mathrm{A} \vee \mathrm{B}) = 1$$
$$P(\mathrm{A}) = \cos^4(\theta/2)$$
$$P(\mathrm{B}) = \sin^4(\theta/2)$$

From the theorem *General Addition* and the probability assignments above it follows that

$$\begin{aligned} P(\mathrm{A} \wedge \mathrm{B}) &= P(\mathrm{A}) + P(\mathrm{B}) - P(\mathrm{A} \vee \mathrm{B}) \\ &= \sin^4(\theta/2) + \cos^4(\theta/2) - 1 \end{aligned}$$

It will suffice to show that the inferred probability $P(\mathrm{B}|\mathrm{A})$ diverges. $P(\mathrm{B}|\mathrm{A})$ is obtained from the inferred probabilities $P(\mathrm{A})$ and $P(\mathrm{A} \wedge \mathrm{B})$ using axiom *Conditional probability*. The resulting equation is then obtained using well-known trigonometric identities.

$$\begin{aligned} P(\mathrm{B}|\mathrm{A}) &= P(\mathrm{A} \wedge \mathrm{B})/P(\mathrm{A}) \\ &= (\sin^4(\theta/2) + \cos^4(\theta/2) - 1)/\cos^4(\theta/2) \\ &= -2\tan^2(\theta/2) \end{aligned}$$

It is clear that $P(\mathrm{B}|\mathrm{A}) \rightarrow -\infty$ as $\theta \rightarrow 0$ from the right. The remaining portrayals (2 and 3) are dealt with below following the approach above but in an abbreviated form.

In Portrayal 2, the following probability assignments are obtained (making appropriate identifications and simplifications).

$$P(\mathrm{A} \wedge \mathrm{B}) = 1$$
$$P(\mathrm{A}) = \cos^4(\theta/2)$$
$$P(\mathrm{B}) = \sin^4(\theta/2)$$

Using *Conditional probability* and the assignments above leads to the following equations.

$$\begin{aligned} P(\mathrm{B}|\mathrm{A}) &= P(\mathrm{A} \wedge \mathrm{B})/P(\mathrm{A}) \\ &= 1/\cos^4(\theta/2). \end{aligned}$$

It is clear $P(\mathrm{B}|\mathrm{A}) \rightarrow +\infty$ as $\theta \rightarrow \pi$ from the left.

In Portrayal 3, the following probability assignments are obtained when $\theta = \pi/2$ (making appropriate identifications and simplifications).

$$P(\mathrm{A} \vee \mathrm{B}) = (1 + \cos(\phi))/2$$
$$P(\mathrm{A} \wedge \neg\mathrm{B}) = (1 + \sin(\phi))/4$$
$$P(\neg\mathrm{A} \wedge \mathrm{B}) = (1 - \sin(\phi))/4$$

Using *Logical equivalence* it follows that

$$P(\mathrm{A} \vee \mathrm{B}) = P(\mathrm{A} \wedge \mathrm{B}) + P(\mathrm{A} \wedge \neg\mathrm{B}) + P(\neg\mathrm{A} \wedge \mathrm{B})$$

$$\Pr(\mathrm{A}) = P(\mathrm{A} \wedge \mathrm{B}) + P(\mathrm{A} \wedge \neg\mathrm{B})$$

Simple algebraic manipulation of the first equation yields

$$P(\mathrm{A} \wedge \mathrm{B}) = P(\mathrm{A} \vee \mathrm{B}) - P(\mathrm{A} \wedge \neg\mathrm{B}) - P(\neg\mathrm{A} \wedge \mathrm{B})$$

These equations are then obtained using *Conditional probability* and algebraic manipulation:

$$\begin{aligned} P(\mathrm{B}|\mathrm{A}) &= P(\mathrm{A} \wedge \mathrm{B})/P(\mathrm{A}) \\ &= 2\cos(\phi)/(2\cos(\phi) + \sin(\phi) + 1) \end{aligned}$$

It is clear that $P(\mathrm{B}|\mathrm{A}) \to +\infty$ as $\phi \to \arccos(-4/5)$ from the right.

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